Week 2: Electronic filters, the mechanical domain, and equivalent circuits

Microphone and Loudspeaker Design - Level 5

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What are we covering today?

1. A quick recap...
2. Electrical filters
3. Equivalent circuits
4. Mechanical domain
5. Impedance analogy
6. Mobility analogy

A weekly fact about Salford..!

Did you know...

- In 1850, under the terms of the Museums Act 1845, the Salford council established the Royal Museum and Public Library; the first unconditional free public library in World! The library sits at the head of Peal park and is now the Salford Museum and Art Gallery.

A quick recap...

AC circuit theory - recap

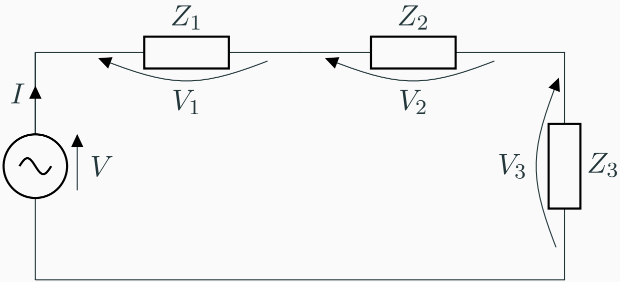

Figure 1: Three arbitrary impedances in series

- For a **series arrangement**, the total impedance Z_T presented by the circuit is,

$$Z_T = Z_1 + Z_2 + Z_3. \tag{1}$$

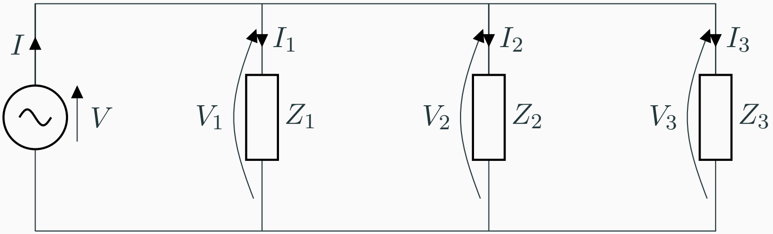

Figure 2: Three arbitrary impedances in parallel

- For a **parallel arrangement**, the total impedance Z_T presented by the circuit is,

$$Z_T = \left(\frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3}\right)^{-1}. \tag{2}$$

- For **two elements** in parallel:

$$Z_T = \frac{Z_1 Z_2}{Z_1 + Z_2}. \tag{3}$$

Voltage and current dividers

- Voltage divider transfer function

$$H = \frac{V_{out}}{V_{in}} = \frac{Z_{out}}{Z_T} \rightarrow \frac{Z_2}{Z_1 + Z_2} \quad (4)$$

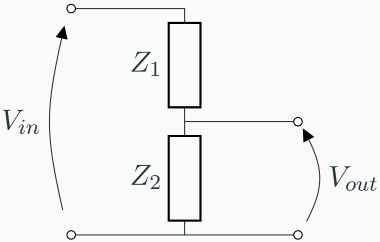

Figure 3: Potential (voltage) divider circuit

- Current divider transfer function

$$H_n = \frac{I_n}{I_T} = \frac{Z_T}{Z_n} \quad (5)$$

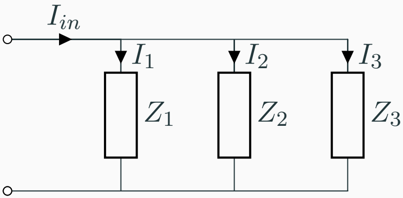

Figure 4: Current divider circuit

Types of component: summary

- Impedance is frequency dependant!
- Resistors

$$Z_{Er} = R \tag{6}$$

- Capacitors

$$Z_{Ec} = \frac{1}{j\omega C} \tag{7}$$

- Inductors

$$Z_{El} = j\omega L \tag{8}$$

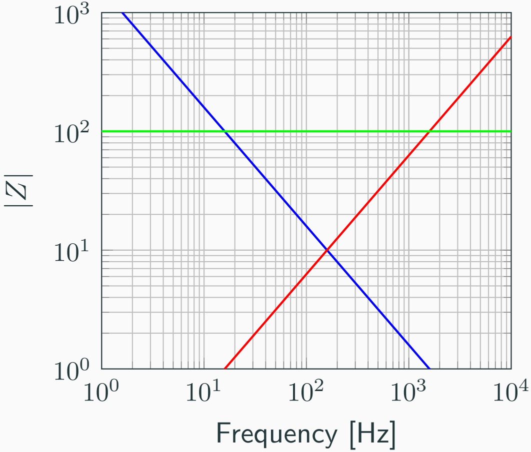

Figure 5: Impedance curves for resistor, capacitor and inductor.

Electrical filters

RC circuit

- Use potential divider rule

$$V_{out} = \frac{Z_C}{Z_R + Z_C} V_{in} \tag{9}$$

- Use component impedances

$$\frac{V_{out}}{V_{in}} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{j\omega RC + 1} \tag{10}$$

- Take magnitude

$$\left| \frac{V_{out}}{V_{in}} \right| = \frac{1}{\sqrt{1 + (\omega RC)^2}} \tag{11}$$

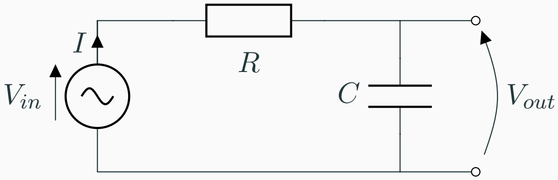

Figure 6: RC circuit

- Transfer function magnitude:

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1+(\omega RC)^2}} \tag{12}$$

- In the limit that $\omega \to 0$

$$\left|\frac{V_{out}}{V_{in}}\right| \to 1 \tag{13}$$

- In the limit that $\omega \to \infty$

$$\left|\frac{V_{out}}{V_{in}}\right| \to 0 \tag{14}$$

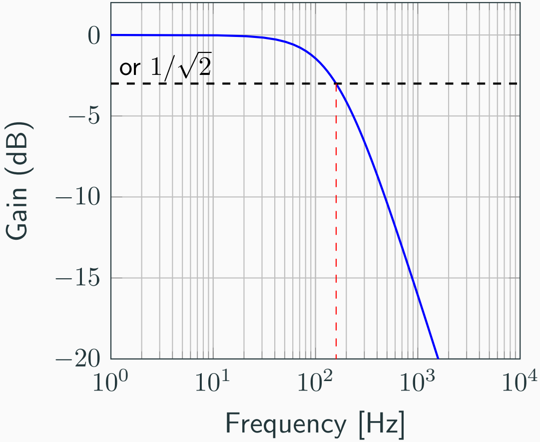

Figure 7: Gain response for an RC circuit, output taken across capacitor.

RC circuit

- The cut-off frequency (ω_c) is when the power output is half that of the input (gain of $1/\sqrt{2}$ or -3dB)

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+(\omega_c RC)^2}}. \tag{15}$$

- This will happen when $(\omega_c RC)^2 = 1$. Hence, we can see that,

$$\omega_c = \frac{1}{RC} \tag{16}$$

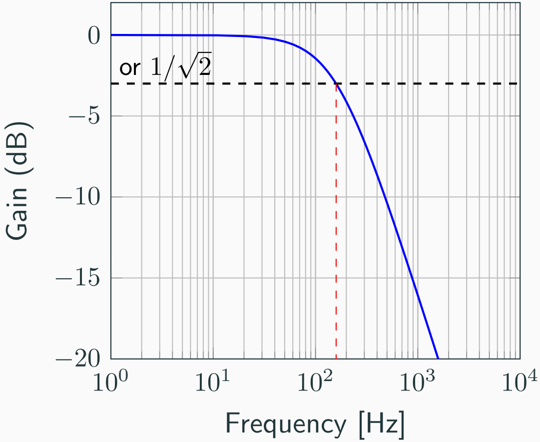

Figure 7: Gain response for an RC circuit, output taken across capacitor.

RC circuit

- Re-parametrise transfer function

$$\frac{V_{out}}{V_{in}} = \frac{1}{j\omega RC + 1} = \frac{1}{j\frac{\omega}{\omega_c} + 1} \quad (17)$$

- The gain at cut off (when $\omega = \omega_c$) is

$$\frac{V_{out}}{V_{in}} = \frac{1}{j+1} = \frac{1-j}{2} = \frac{1}{2} - \frac{j}{2} \quad (18)$$

- Using trigonometry, the phase response at cut-off is,

$$\phi = \tan^{-1}\left(\frac{-0.5}{0.5}\right) \rightarrow -45° \quad (19)$$

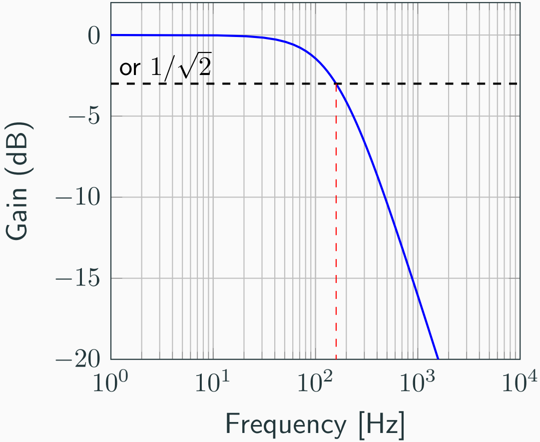

Figure 7: Gain response for an RC circuit, output taken across capacitor.

RL circuit

- Use potential divider rule

$$V_{out} = \frac{Z_L}{Z_R + Z_L} V_{in} \tag{20}$$

- Use component impedances

$$\frac{V_{out}}{V_{in}} = \frac{j\omega L}{R + j\omega L} = \frac{1}{\frac{R}{j\omega L} + 1} \tag{21}$$

- Take magnitude

$$\left| \frac{V_{out}}{V_{in}} \right| = \frac{1}{\sqrt{\left(\frac{R}{\omega L}\right)^2 + 1}} \tag{22}$$

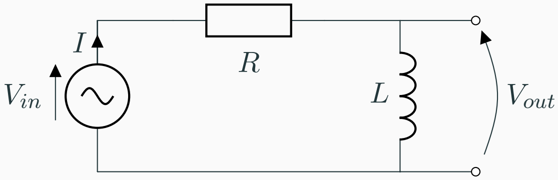

Figure 8: RL circuit

RL circuit

- Transfer function magnitude:

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{\left(\frac{R}{\omega L}\right)^2 + 1}} \tag{23}$$

- In the limit that $\omega \to 0$

$$\left|\frac{V_{out}}{V_{in}}\right| \to 0 \tag{24}$$

- In the limit that $\omega \to \infty$

$$\left|\frac{V_{out}}{V_{in}}\right| \to 1 \tag{25}$$

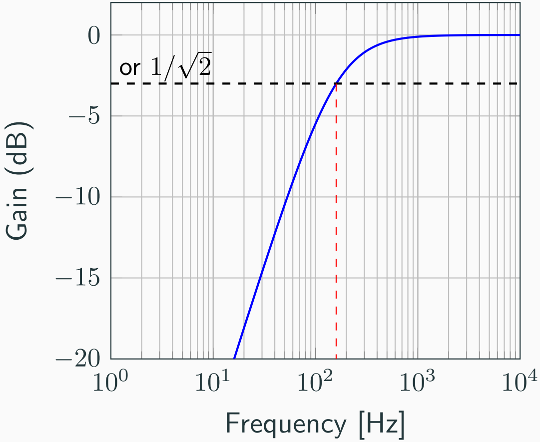

Figure 9: Gain response for an RL circuit, output taken across capacitor.

RL circuit

- The cut-off frequency (ω_c) is when the power output is half that of the input (gain of $1/\sqrt{2}$ or -3dB)

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{\left(\frac{R}{\omega_c L}\right)^2 + 1}}. \quad (26)$$

- This will happen when $\left(\frac{R}{\omega_c L}\right)^2 = 1$. Hence, we can see that,

$$\omega_c = \frac{R}{L}. \quad (27)$$

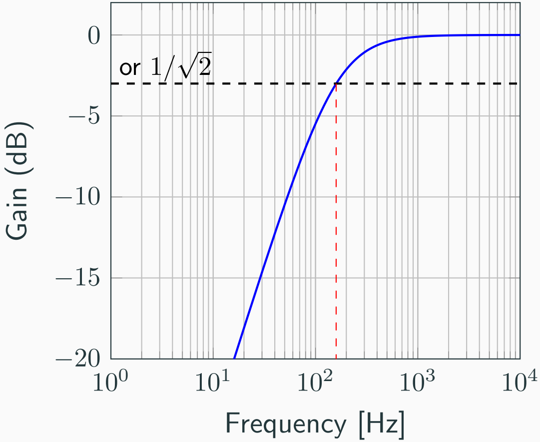

Figure 9: Gain response for an RL circuit, output taken across capacitor.

RL circuit

- Re-parametrise transfer function

$$\frac{V_{out}}{V_{in}} = \frac{1}{\frac{R}{j\omega L} + 1} = \frac{1}{1 - j\frac{\omega_c}{\omega}} \qquad (28)$$

- The gain at cut off (when $\omega = \omega_c$) is

$$\frac{V_{out}}{V_{in}} = \frac{1}{1 - j} = \frac{1}{2} + \frac{j}{2} \qquad (29)$$

- Using trigonometry, the phase response at cut-off is,

$$\phi = \tan^{-1}\left(\frac{0.5}{0.5}\right) \rightarrow 45^\circ. \qquad (30)$$

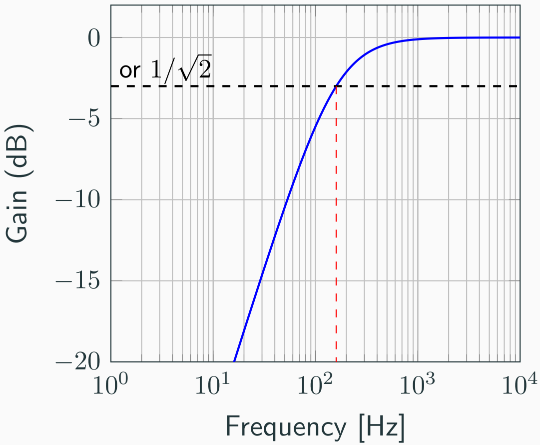

Figure 9: Gain response for an RL circuit, output taken across capacitor.

LCR circuit

- Total impedance

$$Z_{LCR} = R + j\omega L + \frac{1}{j\omega C} \tag{31}$$

- Use potential divider rule

$$V_{out} = \frac{Z_R}{Z_R + Z_L + Z_C} V_{in} \tag{32}$$

- Use component impedances

$$\frac{V_{out}}{V_{in}} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}} = \frac{1}{1 + \frac{j\omega L}{R} + \frac{1}{j\omega RC}} \tag{33}$$

- Take magnitude

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}} \tag{34}$$

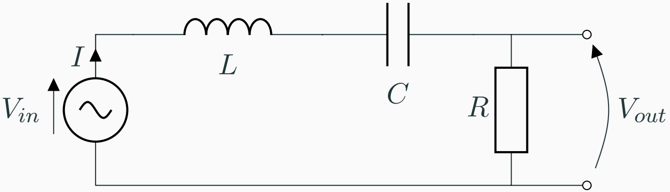

Figure 10: LCR circuit

LCR circuit

- Transfer function magnitude:

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{1}{\sqrt{1 + \left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right)^2}} \quad (35)$$

- In the limit that $\omega \to 0$

$$\left|\frac{V_{out}}{V_{in}}\right| \to 0 \quad \text{(Det. by capacitance)} \quad (36)$$

- In the limit that $\omega \to \infty$

$$\left|\frac{V_{out}}{V_{in}}\right| \to 0 \quad \text{(Det. by inductance)} \quad (37)$$

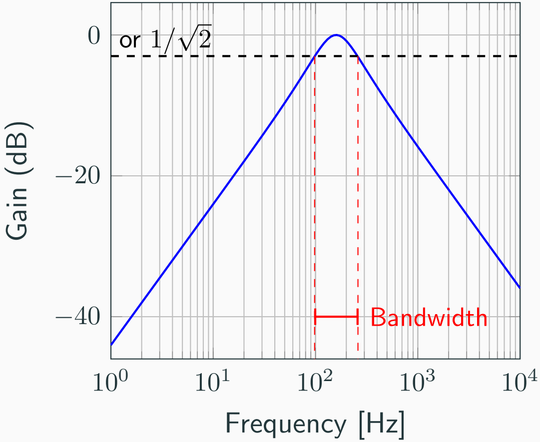

Figure 11: Gain response for an LCR circuit, output taken across resistor.

LCR circuit

- Maximum gain when,

$$\left(\frac{\omega L}{R} - \frac{1}{\omega RC}\right) = 0 \qquad (38)$$

- Rearrange to find resonant frequency

$$\omega_r = \frac{1}{\sqrt{LC}} \qquad (39)$$

- The BW and resonant frequency determine the Q-Factor

$$Q = \frac{\omega_r}{BW} = \frac{1}{\sqrt{LC}}\frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}} \qquad (40)$$

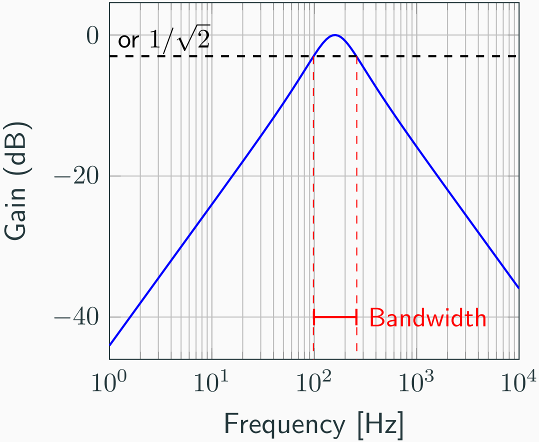

Figure 11: Gain response for an LCR circuit, output taken across resistor.

Impedance: a common language

- **Electrical impedance** is the measure of the opposition that a circuit presents to a current when a voltage is applied

$$Z_E = \frac{V}{I} \tag{41}$$

- **Mechanical impedance** is a measure of how much a structure resists motion (velocity) when subjected to a force

$$Z_M = \frac{F}{u} \tag{42}$$

- **Acoustic impedance** is a measure of the opposition that a system presents to the acoustic flow (volume velocity) when subjected to acoustic pressure

$$Z_A = \frac{p}{U} \tag{43}$$

Impedance: resistance vs. reactance

- Impedance is generally a complex quantity. It has a real part and an imaginary part.

$$Z = R + jX \tag{44}$$

  - Real part is called the resistance - R
  - Imaginary part is called reactance - X
- Resistance describes energy loss (e.g. through heat, friction, etc.)
- Reactance describes energy storage (e.g. through mag/elec fields, momentum, etc.)

Equivalent circuits

Equivalent circuits: drop and flow

- We have covered 3 key electrical components: resistor, capacitor, inductor.
- There are three generic quantities:
  1. The (voltage) drop across the (electrical) component
  2. The (current) flow through the (electrical) component
  3. The magnitude of the (electrical) component itself (resistance, capacitance, inductance)
- These generic quantities are not limited to electrical components:
  - For a mechanical system we have force (F) and velocity (u)
  - For an acoustic system we have pressure (p) and volume velocity (U)
  - But which quantity is the drop and which is the flow? - **Depends on the problem!**

Mechanical domain

Mechanical domain: three main components

- Three main components: mass elements, springs, and dampers

- Mechanical impedance:

$$Z_M = \frac{F}{u} \tag{45}$$

- Turn out to be very similar to those of electrical components discussed so far.

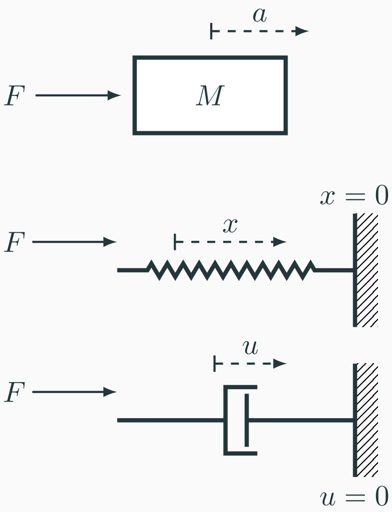

Figure 12: Mass, spring and damper.

Mechanical domain: mass

- Newton's 2nd Law

$$F = Ma \tag{46}$$

- Assume periodic force $F = F_o e^{j\omega t}$

$$F = M\frac{du}{dt} = j\omega M u \tag{47}$$

- The impedance is then,

$$Z_M = \frac{F}{u} = j\omega M \tag{48}$$

- Impedance is proportional to frequency and complex - what electrical component does this look like?

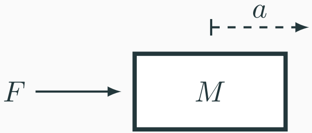

Figure 13: Mass element.

Mechanical domain: spring

- Hooke's Law

$$F = kx = \frac{1}{C}x \tag{49}$$

- Assume periodic force $F = F_o e^{j\omega t}$

$$F = \frac{1}{C}\int u dt = \frac{1}{j\omega C}u \tag{50}$$

- The impedance is then,

$$Z_C = \frac{F}{u} = \frac{1}{j\omega C} \tag{51}$$

- Impedance is inv. prop. to freq. and complex - what elec. component does this look like?

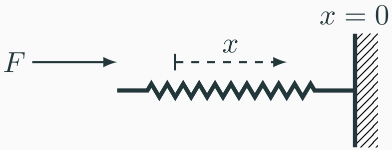

Figure 14: Spring element.

Mechanical domain: damper

- Viscous damping element

$$F = Ru \tag{52}$$

- The impedance is then,

$$Z_R = \frac{F}{u} = R \tag{53}$$

- Impedance is constant wrt frequency and real - what electrical component does this look like?

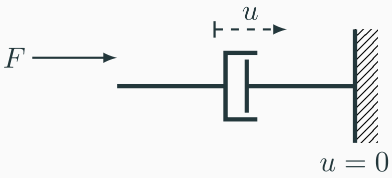

Figure 15: Damping element.

Impedance analogy

Equivalent circuits: drop and flow

- For the **impedance analogy** we think of:
    - Force as being analogous to voltage $F \rightarrow V$
    - Velocity as being analogous to current $u \rightarrow I$

- By drawing this particular equivalence we preserve the analogy between mechanical and electrical impedance:

$$Z_M \rightarrow Z_E \tag{54}$$

- But, the topology of our problem is lost... i.e. mechanical system is arranged differently to its analogous electrical circuit
- Another popular one is called the mobility analogy...

Impedance analogy: summary

Element	Impedance analogy	Mobility analogy
Mass	Mass $\leftrightarrow$ Inductor $Z_M = j\omega M_M \leftrightarrow Z_E = j\omega L_E$	
Spring	Spring $\leftrightarrow$ Capacitor $Z_M = \frac{1}{j\omega C_M} \leftrightarrow Z_E = \frac{1}{j\omega C_E}$	
Damper	Damper $\rightarrow$ Resistor $Z_M = R_M \leftrightarrow Z_E = R_E$	

Impedance analogy: mass-spring-damper

- Use analogy between mechanical and electrical components to model mechanical systems as electric circuits.
- To draw equivalent circuit first recall the definition of impedance analogy:

$$F \to V \qquad u \to I \tag{55}$$

- Note that the mass, spring and damper all have the same velocity, because they are connected together...

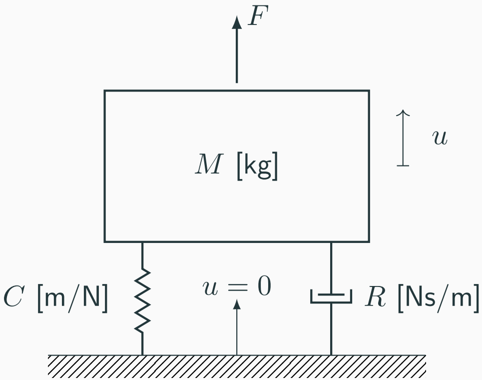

Figure 16: Mass-spring-damper.

Impedance analogy: mass-spring-damper

- Using AC circuit theory we can easily calculate the impedance of the mechanical system,

$$Z_M = j\omega M_M + \frac{1}{j\omega C_M} + R_M \tag{56}$$

- Mechanical velocity given by,

$$u = \frac{F}{Z_M} = \frac{F}{j\omega M_M + \frac{1}{j\omega C_M} + R_M} \tag{57}$$

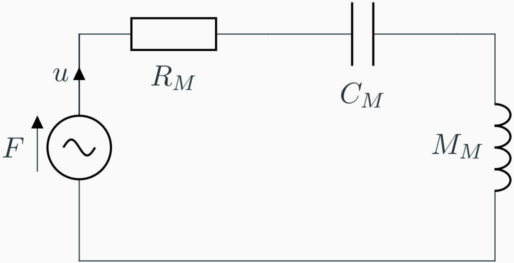

Figure 17: Mass-spring-damper equivalent circuit

Impedance analogy: mass-spring-damper

- Using equivalent circuit we can calculate the velocity of the mass,

$$u = \frac{F}{j\omega M_M + \frac{1}{j\omega C_M} + R_M} \quad (58)$$

- As expected, the response looks just like an LCR circuit!

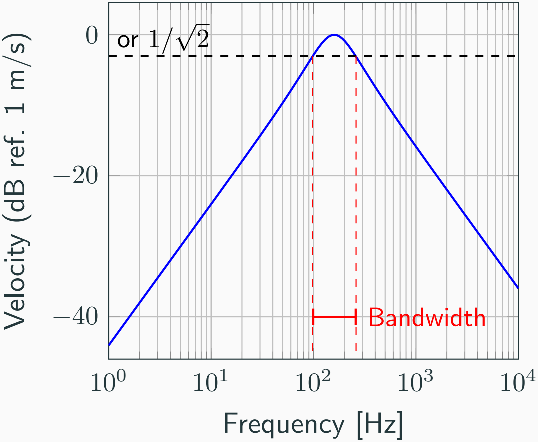

Figure 11: Velocity response of a mass-spring system.

Mobility analogy

Mobility analogy

- For the **impedance analogy** we made the following equivalences:
  - Force as being analogous to voltage $F \to V$ (drop parameter)
  - Velocity as being analogous to current $u \to I$ (flow parameter)

- But there is no reason why we cant consider the opposite!

- For the **mobility analogy** we make the following equivalences:
  - Force as being analogous to current $F \to I$ (flow parameter)
  - Velocity as being analogous to voltage $u \to V$ (drop parameter)

Next week...

- Mobility analogy
- Impedance stuff
- Q-factor

- Reading:
  - Mechanical domain: lecture notes, chp. 4, pg. all
  - Impedance and mobility analogies: lecture notes, chp. 4 pg. all